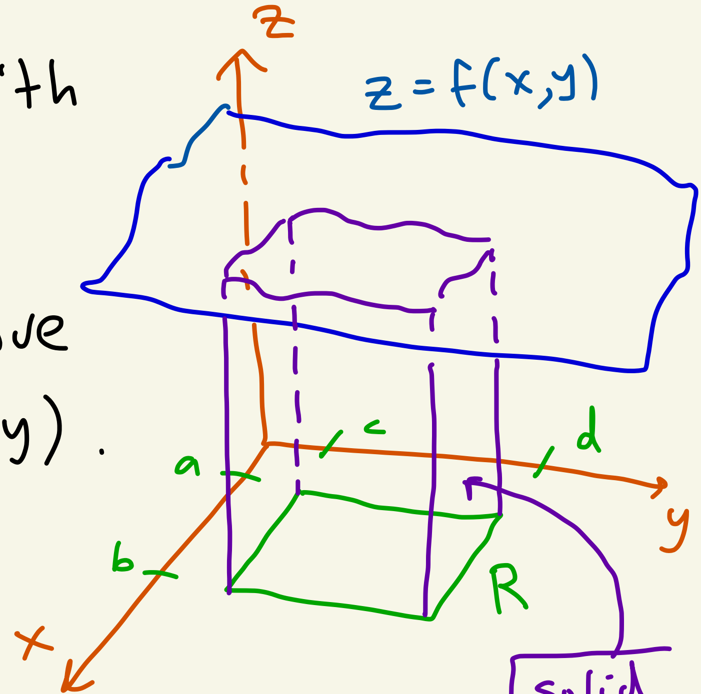
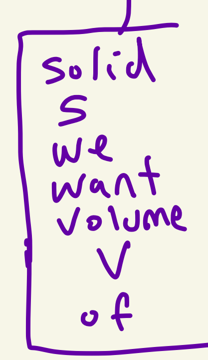
Topic 5 - Double Integrals

Part 1 – Double Integrals over rectangles

Consider a function $f(x,y)$ defined on a closed rectangle R defined by all x,y with $a \le x \le b$ and $c \le x \le d$.
Suppose for now that $f(x,y) \geq 0$ everywhere.





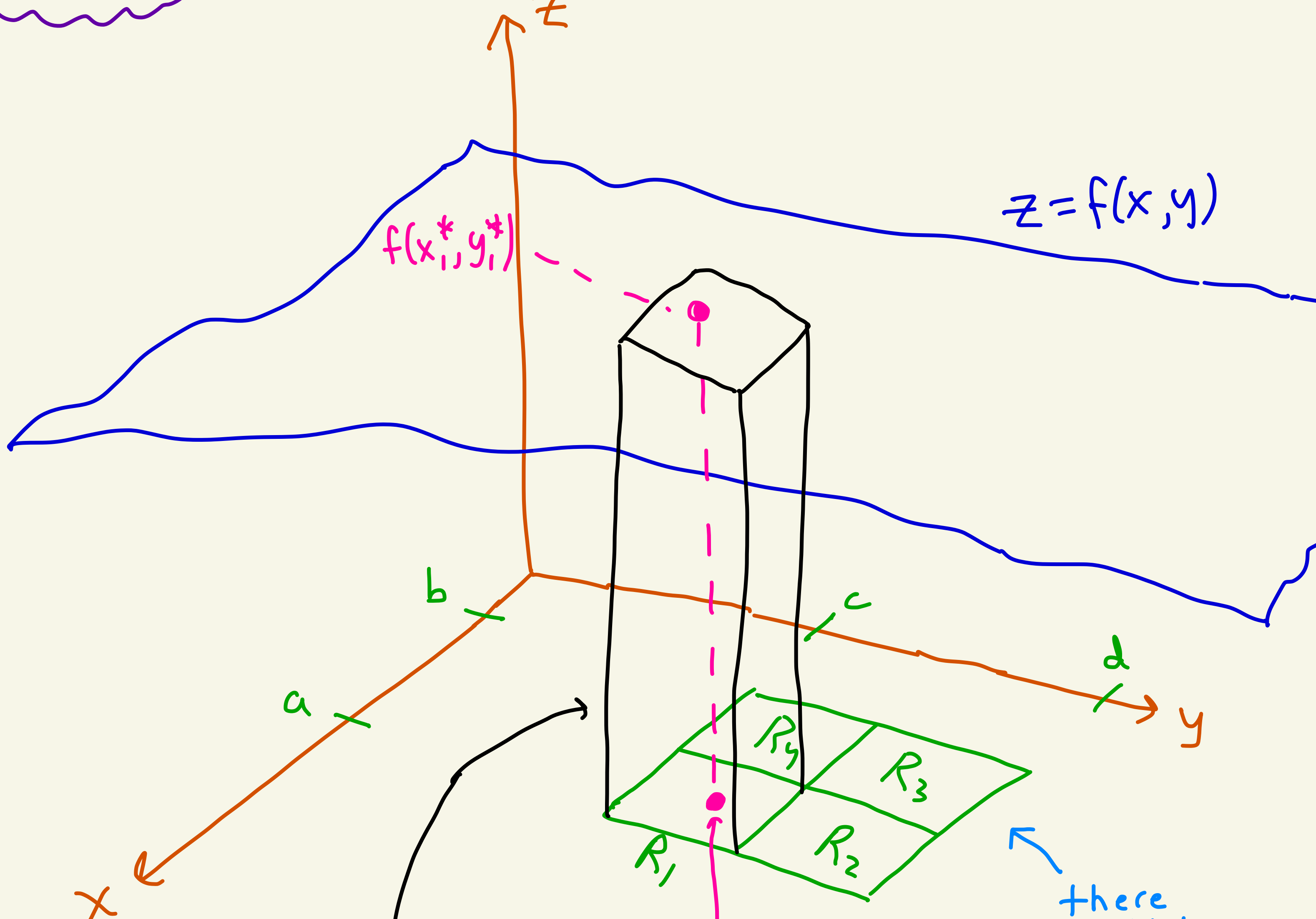
We want to come up with a way to find the volume V of the solid S that lies above R and under $z = f(x,y)$.

A partition of R is formed by subdividing R into n rectangular subregions $R_1, R_2, \ldots, R_n$ using lines parallel to the x-axis or y-axis. A rectangle R_k has side lengths Δx_k and Δy_k, and area $\Delta A_k = \Delta x_k \Delta y_k$. For each k let (x_k^*, y_k^*) be an arbitrary point in R_k.

Then, the volume V is approximated by

$$V \approx \sum_{k=1}^{n} \underbrace{\underbrace{f(x_k^*, y_k^*)}_{\text{height of box above } R_k} \; \underbrace{\Delta A_k}_{\text{area of base of box } R_k}}_{f(x_k^*, y_k^*)\Delta A_k \text{ is volume of box above } R_k}$$

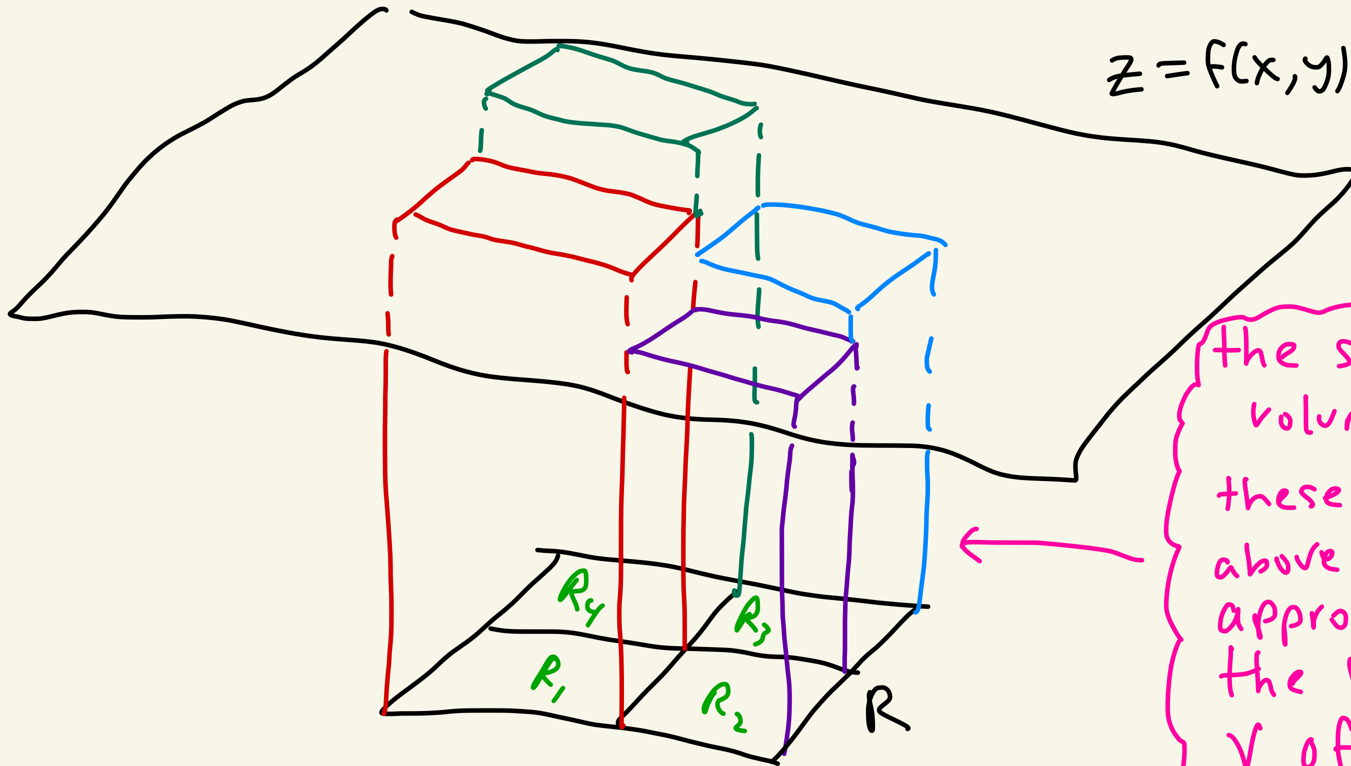
add up the volumes of all n boxes



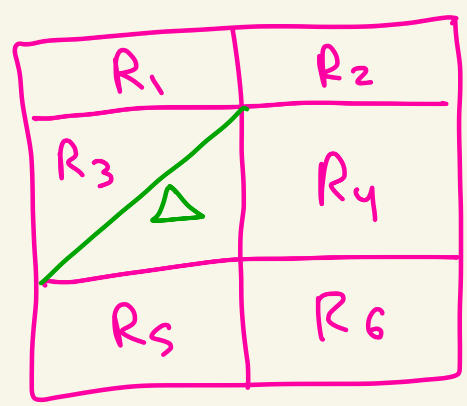
this box has area $f(x_1^*, y_1^*)\Delta A_1$

(x_1^*, y_1^*)

there would be 3 more boxes, one above each of these R_k



the sum of the volumes of these boxes above the R_k approximates the volume V of S



If Δ is the maximum length of the diagonals of all the R_k's, then

as $\underbrace{n \to \infty}_{\text{more and more boxes}}$ and $\underbrace{\Delta \to 0}_{\text{sizes of bases } R_k \text{ of boxes get smaller and smaller}}$

then $\underbrace{\Delta A_k \to 0}_{\text{areas of bases } R_k \text{ go to } 0.}$.

We now make better and better approximations to the volume of the solid under $z = f(x,y)$ by letting $n \to 0$ and $\Delta \to 0$.

DEF: The double integral of f over R is defined to be

$$\underbrace{\iint_R f(x,y)\,dA}_{\text{notation for integral}} = \underbrace{\lim_{\substack{\Delta \to 0 \\ n \to \infty}}}_{\text{limit of more and more } R_k\text{'s of smaller and smaller sizes}} \underbrace{\sum_{k=1}^{n} f(x_k^*, y_k^*)\,\Delta A_k}_{\text{sum of the volumes of the boxes in our subdivision}}$$

if the limit exists.

Notes:

- If $f(x,y) \geq 0$ over R, then we can define the volume V to be the integral if it exists.
- If f is also negative over R you get a net volume of positive and negative boxes.
- If f is continuous over R then the integral exists.

Q: How do we calculate the integral?

A: With iterated integrals.

Theorem: If f is continuous on R defined by $a \leq x \leq b$ and $c \leq y \leq d$ then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

you can switch the order of x and y and you get the same answer

where the above means:

the c-d bounds go with y

the a-b bounds go with x

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \underbrace{\left(\int_a^b f(x,y)\,dx \right)}_{\text{integrate with respect to } x \text{ first}} dy$$

then integrate with respect to y

and

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

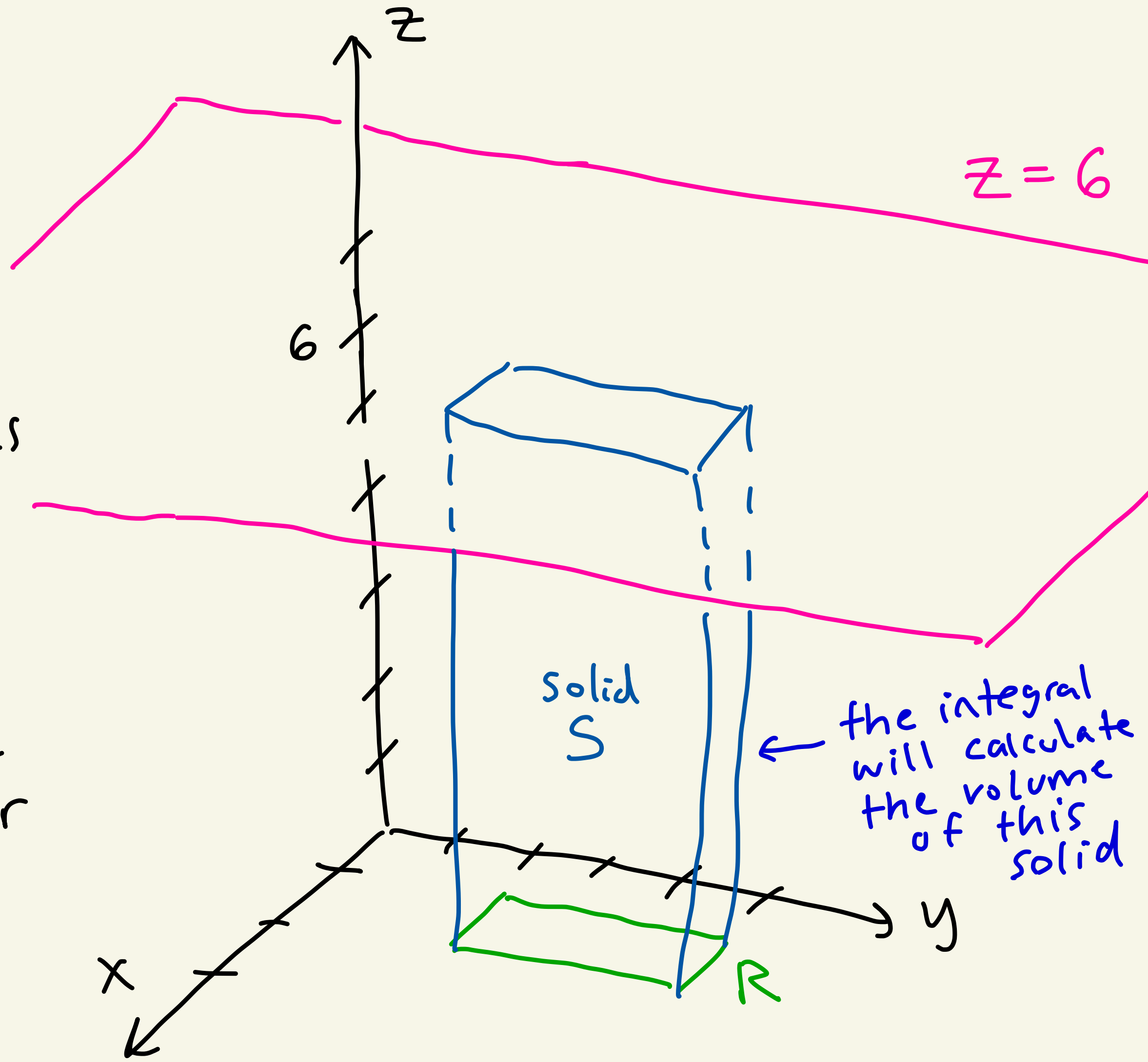
c,d go with y

a,d go with x

Ex: Find $\iint_R 6 \, dA$ where R is defined by $1 \leq x \leq 2$ and $2 \leq y \leq 5$

Here the function is $f(x,y) = 6$ and describes the plane $z = 6$

We want the volume of the solid S under $z = 6$ and above R.



$$\iint_R 6 \, dA = \int_2^5 \underbrace{\left(\int_1^2 6 \, dx \right)}_{\text{integrate with respect to } x} dy$$

$$= \int_2^5 \left(6x \Big|_{x=1}^{2} \right) dy$$

$$= \int_2^5 \left(6(2) - 6(1) \right) dy$$

$$= \underbrace{\int_2^5 6 \, dy}_{\text{integrate with respect to } y}$$

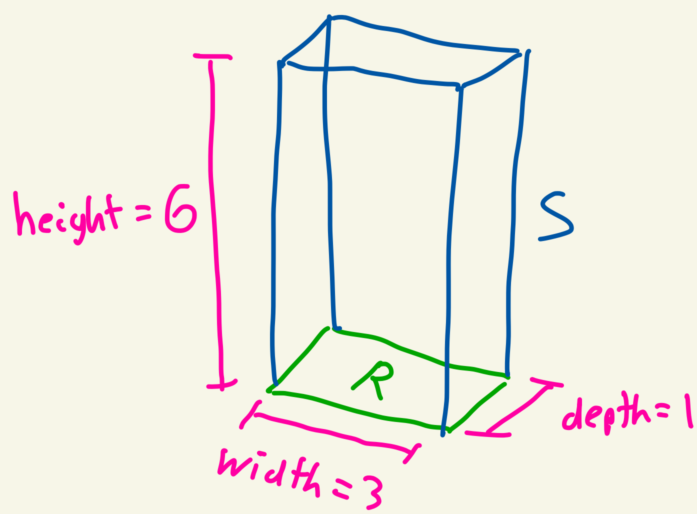
$$= 6y \Big|_{y=2}^{5}$$

$$= 6(5) - 6(2)$$

$$= 30 - 12$$

$$= \boxed{18}$$

The integral agrees with our usual idea of the volume of S as $V = \underbrace{(1)}_{\text{depth}} \cdot \underbrace{(3)}_{\text{width}} \cdot \underbrace{(6)}_{\text{height}} = 18$

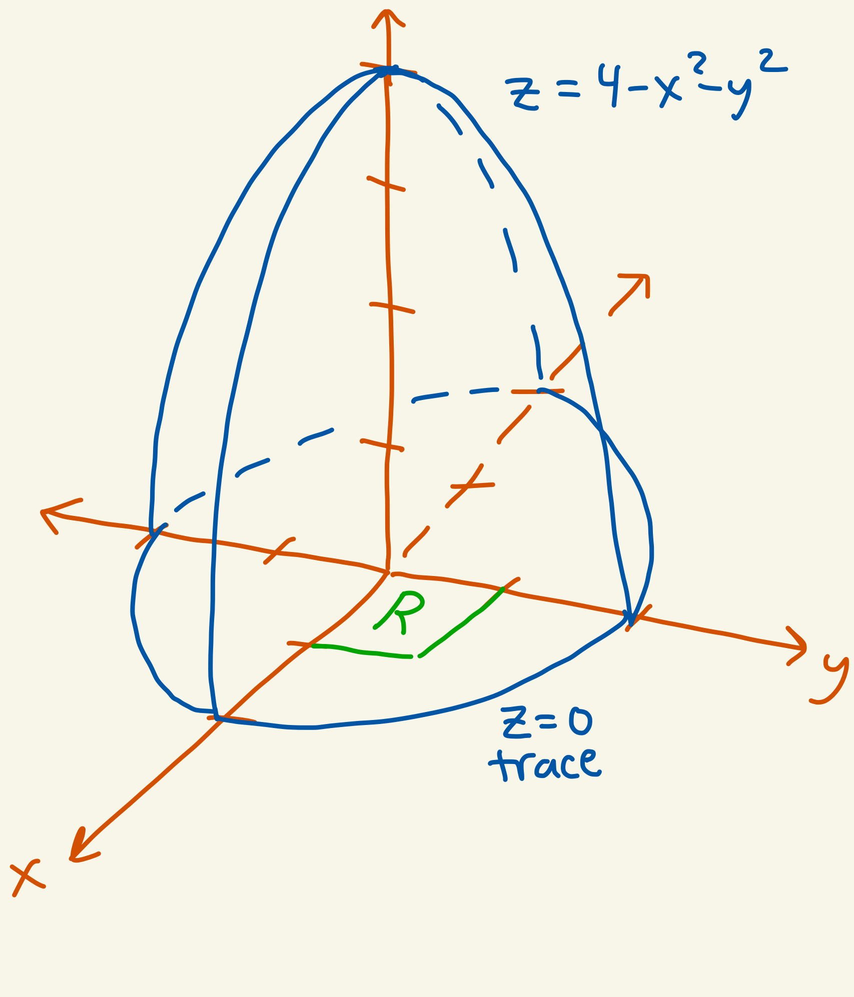


Ex: Find the volume of the solid S that lies under the paraboloid $x^2 + y^2 + z = 4$ and above the rectangle R defined by $0 \le x \le 1$ and $0 \le y \le 1$.

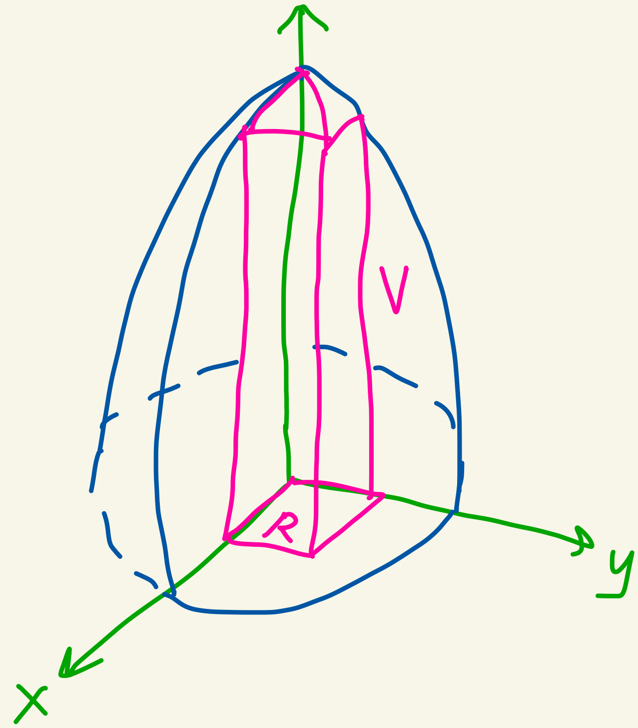
The surface is $z = 4 - x^2 - y^2$.

Trace at $z=0$:
$x^2 + y^2 = 0$

When $x=0, y=0$ we get $z=4$



The integral represents this volume



R given by
$0 \le x \le 1$
$0 \le y \le 1$

We get

$$\iint_R (4-x^2-y^2)\,dA = \int_0^1 \left[\int_0^1 \underbrace{(4-x^2-y^2)\,dx}_{\text{integrate with respect to } x \text{ and treat } y \text{ as a constant}} \right] dy$$

$$= \int_0^1 \underbrace{\left[\left(4x - \frac{x^3}{3} - y^2 x \right) \Bigg|_{x=0}^{1} \right]}_{\text{next plug in } x=1 \text{ and } x=0 \text{ and subtract}} dy$$

$$= \int_0^1 \left[\underbrace{\left(4(1) - \frac{(1)^3}{3} - y^2(1)\right)}_{\substack{x=1 \\ \text{term}}} - \underbrace{\left(4(0) - \frac{0^3}{3} - y^2(0)\right)}_{\substack{x=0 \\ \text{term}}} \right] dy$$

$$= \int_0^1 \left[\left(\frac{11}{3} - y^2\right) - 0 \right] dy$$

$$= \underbrace{\int_0^1 \left(\frac{11}{3} - y^2\right) dy}_{\text{integrate with respect to } y}$$

$$= \frac{11}{3} y - \frac{y^3}{3} \Big|_0^1$$

$$= \underbrace{\left(\frac{11}{3}(1) - \frac{(1)^3}{3}\right)}_{\substack{y=1 \\ \text{term}}} - \underbrace{\left(\frac{11}{3}(0) - \frac{0^3}{3}\right)}_{\substack{y=0 \\ \text{term}}}$$

$$= \left(\frac{11}{3} - \frac{1}{3}\right) - (0)$$

$$\boxed{= \frac{10}{3} = 3.33\overline{3}}$$

Answer

Ex: Calculate $\iint_R \sqrt{\frac{x}{y}}\, dA$ where

R is the rectangle defined by $0 \le x \le 1,\ 1 \le y \le 4.$

Note that

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}} = \frac{x^{1/2}}{y^{1/2}} = x^{1/2} \cdot \frac{1}{y^{1/2}} = x^{1/2} \cdot y^{-1/2}$$

Thus,

$$\iint_R \sqrt{\frac{x}{y}}\, dA = \int_0^1 \Big(\underbrace{\int_1^4 x^{1/2} y^{-1/2}\, dy}_{\text{integrate with respect to } y \text{, treat } x \text{ as a constant}} \Big)\, dx$$

$$= \int_0^1 \left(x^{1/2} \cdot \frac{y^{-1/2+1}}{-1/2+1} \Bigg|_{y=1}^{4} \right) dx$$

$$= \int_0^1 \underbrace{\left(x^{1/2}\, 2y^{1/2} \Bigg|_{y=1}^{4} \right)}_{\text{plug in } y=4 \text{ and } y=1 \text{ and subtract}} dx$$

$$= \int_0^1 \left[x^{1/2} \cdot 2 \cdot \underbrace{4^{1/2}}_{\sqrt{4}} - x^{1/2} \cdot 2 \cdot \underbrace{1^{1/2}}_{\sqrt{1}=1} \right] dx$$

$$= \int_0^1 \left(4x^{1/2} - 2 \cdot x^{1/2} \right) dx$$

$$= \int_0^1 2x^{1/2} \, dx$$

$$= 2 \frac{x^{3/2}}{(3/2)} \Big|_0^1$$

$$= 2 \cdot \frac{2}{3} x^{3/2} \Big|_0^1$$

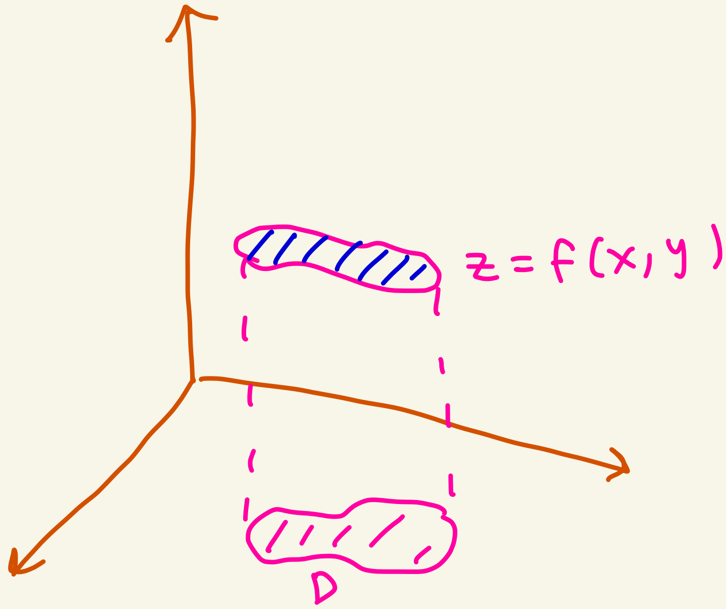
$$= \frac{4}{3} \cdot \left[1^{3/2} - 0^{3/2} \right]$$

$$= \frac{4}{3} \left[1 - 0 \right]$$

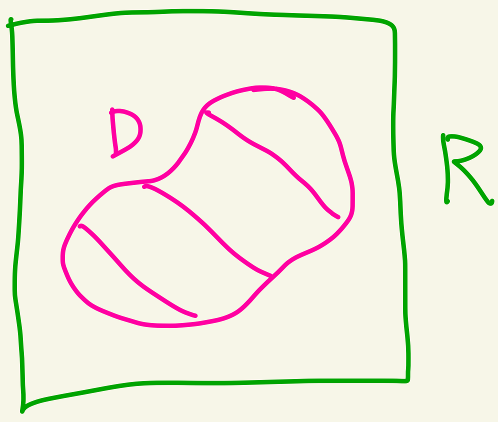
$$= \boxed{4/3} \longleftarrow \boxed{\text{Answer}}$$

Part 2 – Double Integrals over general regions

Suppose we want to define $\iint_D f(x,y)\,dA$ for some f defined on D.



Suppose D is bounded, that is we can encapsulate D inside a rectangle R.



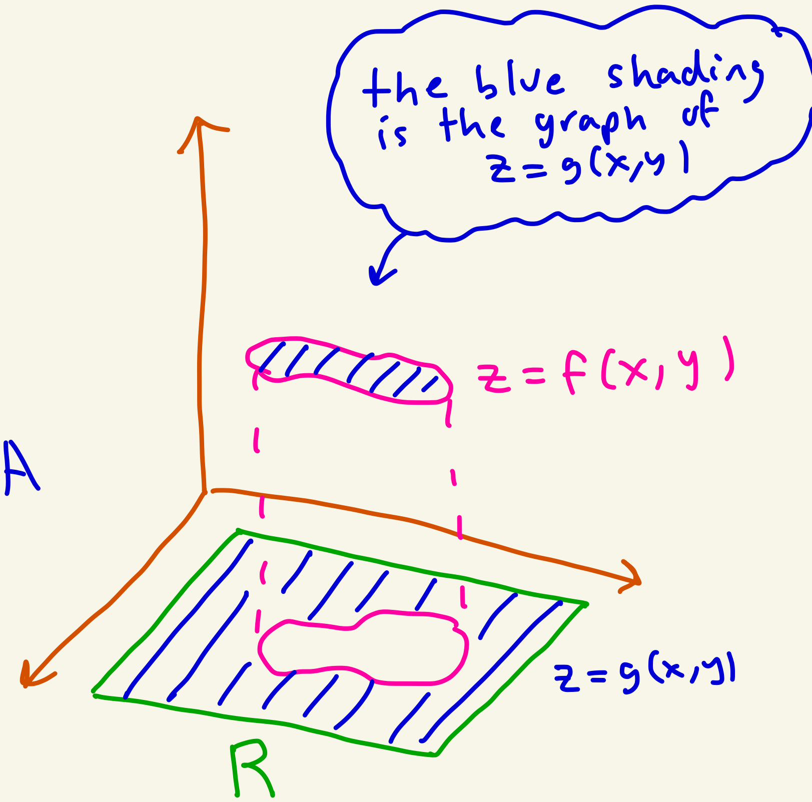
Define a new function g on R where $g(x,y)$ is equal to $f(x,y)$ at the points (x,y) in D, and $g(x,y)=0$ at the points (x,y) outside of D.

That is,

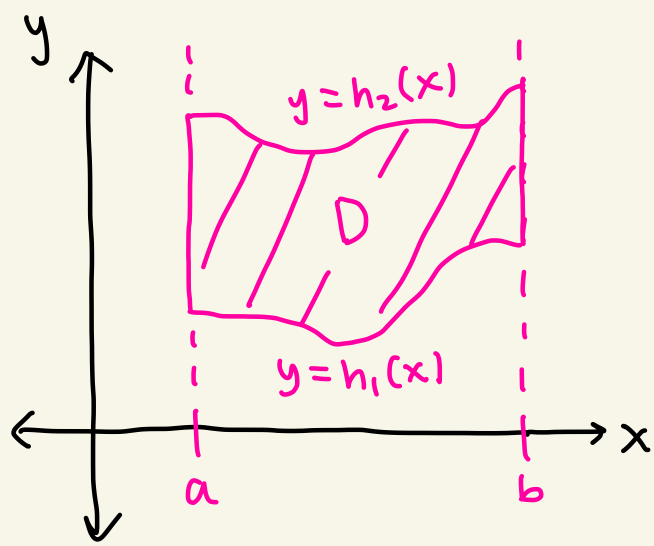
$$g(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \text{ is in } D \\ 0, & \text{if } (x,y) \text{ is not in } D \end{cases}$$

Define the double integral of f over D to be:

$$\iint_D f(x,y)\,dA = \iint_R g(x,y)\,dA$$

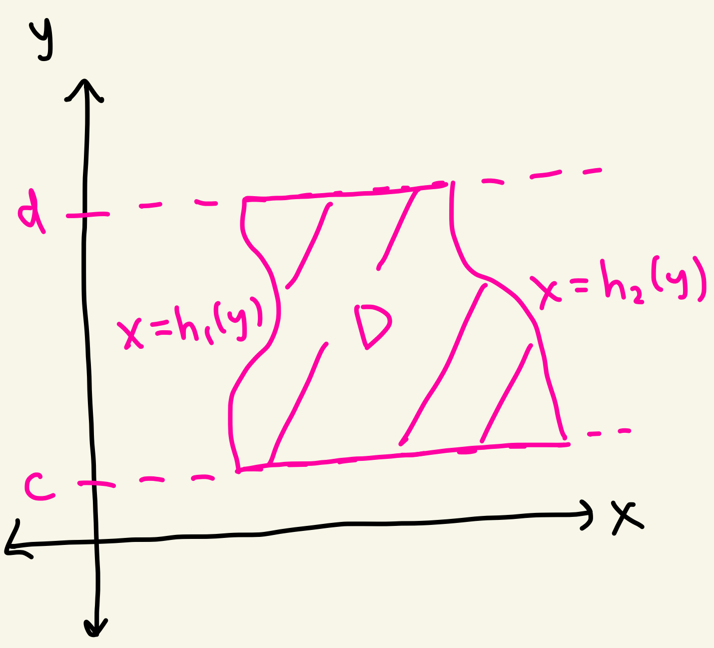


How do we compute the above.
Here are two scenarios.



If D is defined by $a \le x \le b$ and $h_1(x) \le y \le h_2(x)$ then

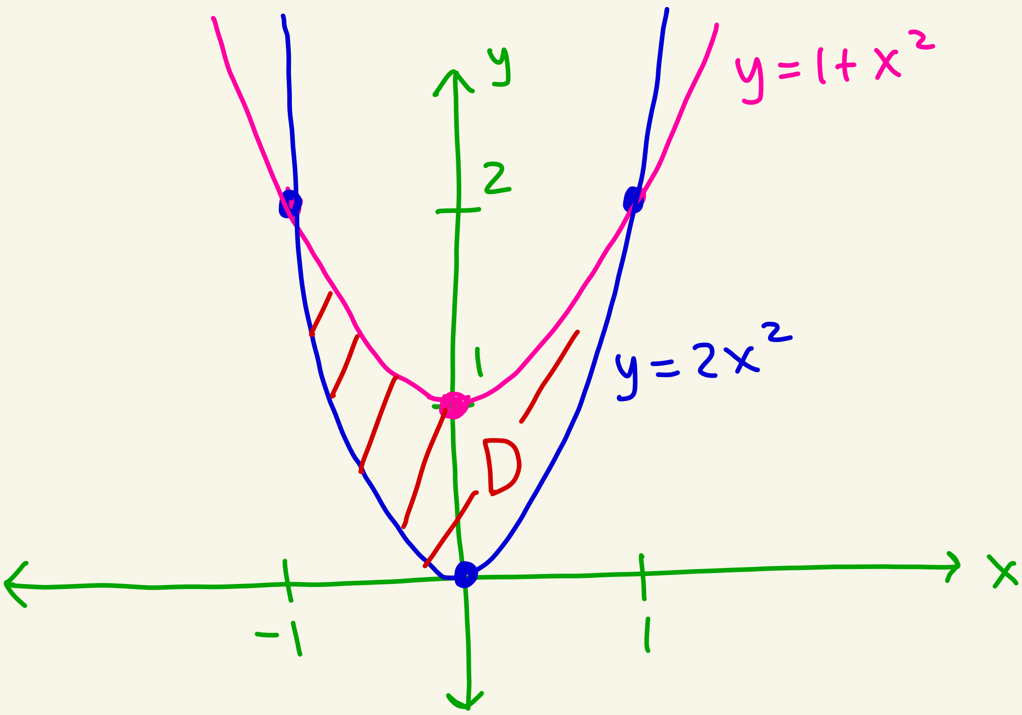
$$\iint_D f(x,y)\,dA = \int_a^b \int_{h_1(x)}^{h_2(x)} f(x,y)\,dy\,dx$$

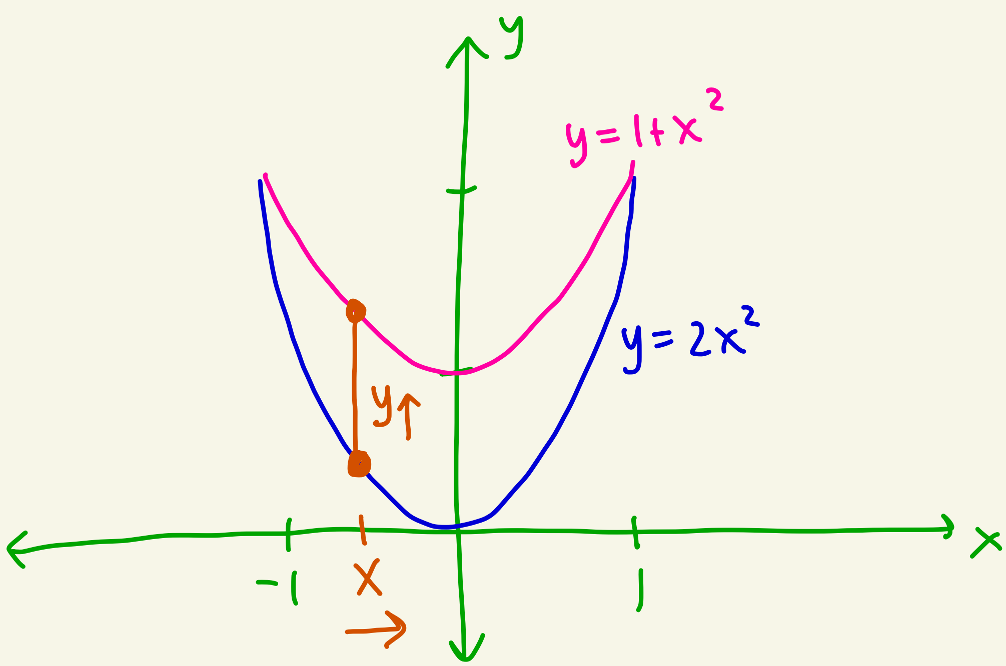


If D is defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$ then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Ex: Compute $\iint_D (x+2y)\,dA$ where D is the region bounded by $y=2x^2$ and $y=1+x^2$





We have

$$\iint_D (x+2y)\,dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx$$

$$= \int_{-1}^{1}\left[xy + y^2 \Big|_{2x^2}^{1+x^2} \right] dx$$

$$= \int_{-1}^{1}\Big[\underbrace{\big(x(1+x^2) + (1+x^2)^2\big)}_{x+x^3+1+2x^2+x^4} - \underbrace{\big(x(2x^2) + (2x^2)^2\big)}_{-2x^3-4x^4} \Big] dx$$

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\,dx$$

$$= \left(-\frac{3}{5}x^5 - \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + x\right)\Bigg|_{-1}^{1}$$

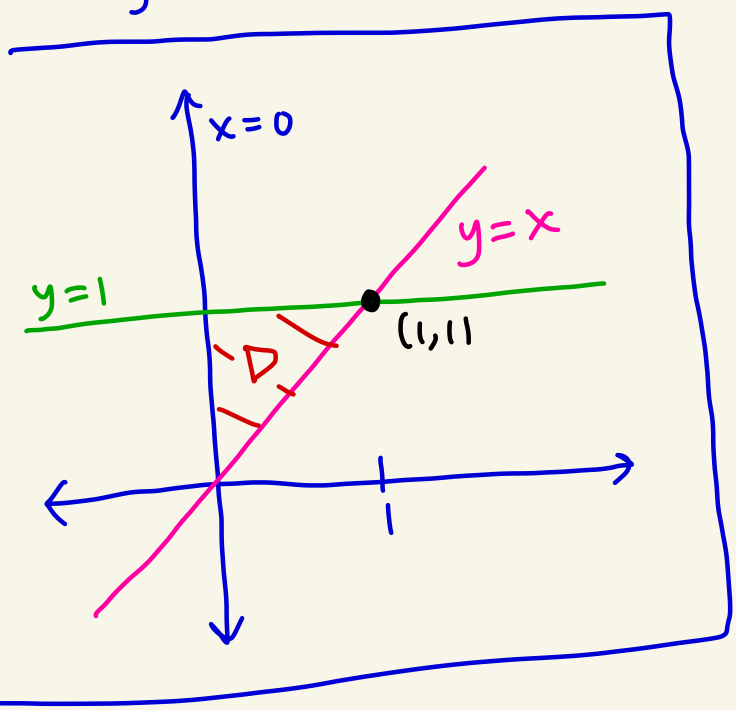
$$= \left(-\frac{3}{5}(1)^5 - \frac{1}{4}(1)^4 + \frac{2}{3}(1)^3 + \frac{1}{2}(1)^2 + (1)\right)$$
$$- \left(-\frac{3}{5}(-1)^5 - \frac{1}{4}(-1)^4 + \frac{2}{3}(-1)^3 + \frac{1}{2}(-1)^2 + (-1)\right)$$

$$= -\frac{3}{5} - \cancel{\frac{1}{4}} + \frac{2}{3} + \cancel{\frac{1}{2}} + 1 - \frac{3}{5} + \cancel{\frac{1}{4}} + \frac{2}{3} - \cancel{\frac{1}{2}} + 1$$

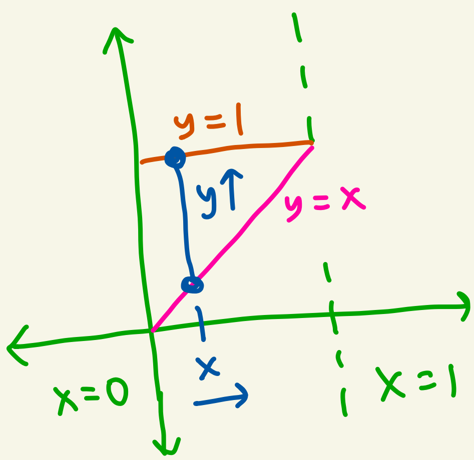
$$= -\frac{6}{5} + \frac{4}{3} + 2 = \frac{-18+20+30}{15} = \boxed{\frac{32}{15}}$$

Ex: Evaluate $\displaystyle\iint_D \sin(y^2)\,dy\,dx$

where D is the triangle bounded by $x=0$, $y=1$, and $y=x$.



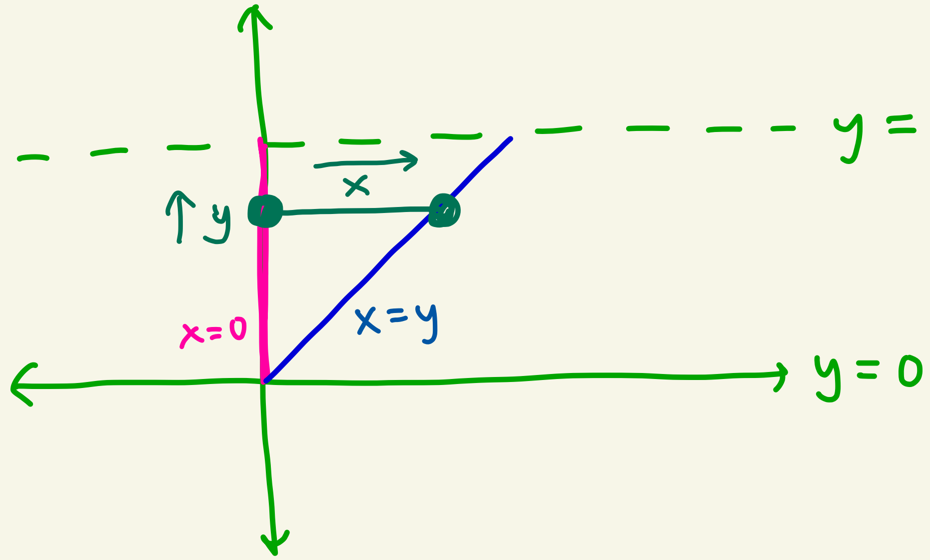
Attempt 1:



$$\underbrace{\int_0^1 \int_x^1 \sin(y^2)\,dy}\,dx$$

can't do since what is the antiderivative of $\sin(y^2)$?

Attempt 2:



$$\int_0^1 \left(\int_0^y \sin(y^2)\,dx \right) dy = \int_0^1 \left(x \sin(y^2) \Big|_{x=0}^{y} \right) dy$$

$$= \int_0^1 \left[y \sin(y^2) - 0 \cdot \sin(y^2) \right] dy$$

$$= \int_0^1 y \sin(y^2)\,dy = \int_0^1 \frac{1}{2} \sin(u)\,du = -\frac{1}{2}\cos(u) \Big|_{u=0}^{1}$$

$u = y^2$
$du = 2y\,dy$
$\frac{1}{2} du = y\,dy$
$y = 0 \to u = 0^2 = 0$
$y = 1 \to u = 1^2 = 1$

$$= -\frac{1}{2}\cos(1) - \left(-\frac{1}{2} \underbrace{\cos(0)}_{1} \right)$$

$$= -\frac{1}{2}\cos(1) + \frac{1}{2}$$

$$= \boxed{\frac{1}{2} - \frac{1}{2}\cos(1)}$$

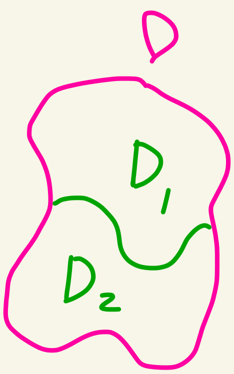
Some properties of the integral:

① $$\iint_D [f(x,y)+g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$$

② $$\iint_D c\,f(x,y)\,dA = c\iint_D f(x,y)\,dA$$ when c is a constant

③ If D is the union of D_1 and D_2 where D_1 and D_2 have no overlap then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

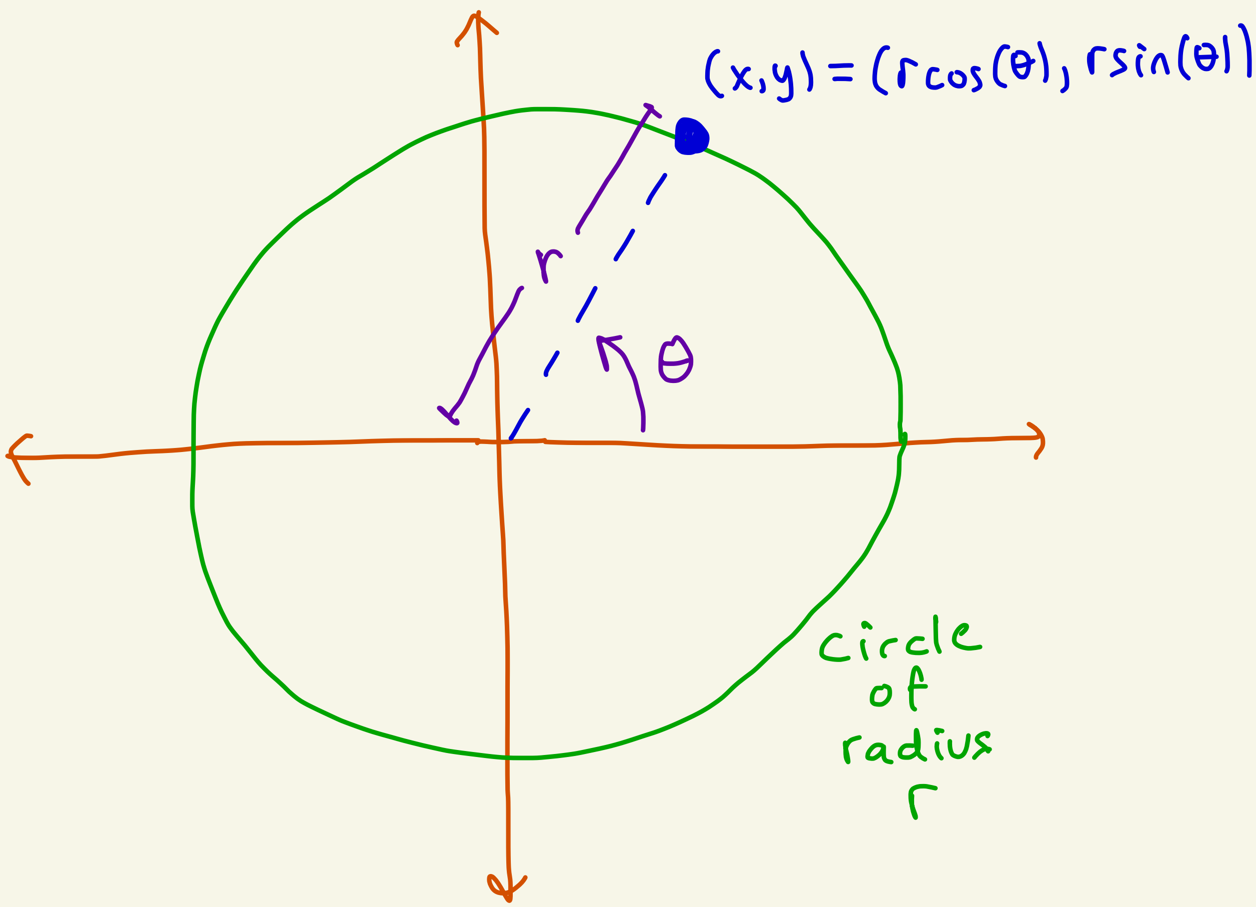


④ $$\iint_D 1\,dA = \text{area of } D$$

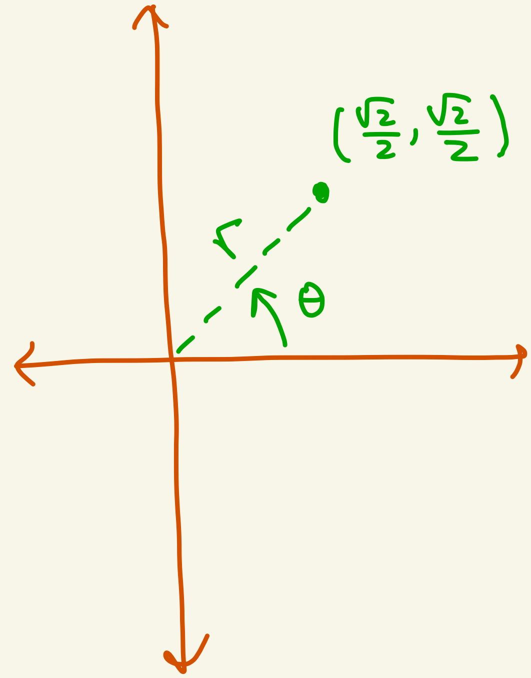
Part 3 – Polar Coordinates

Recall that polar coordinates are given by:

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$x^2 + y^2 = r^2$$
$$\tan(\theta) = y/x$$



Ex: Find the polar coordinates for $(x,y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$



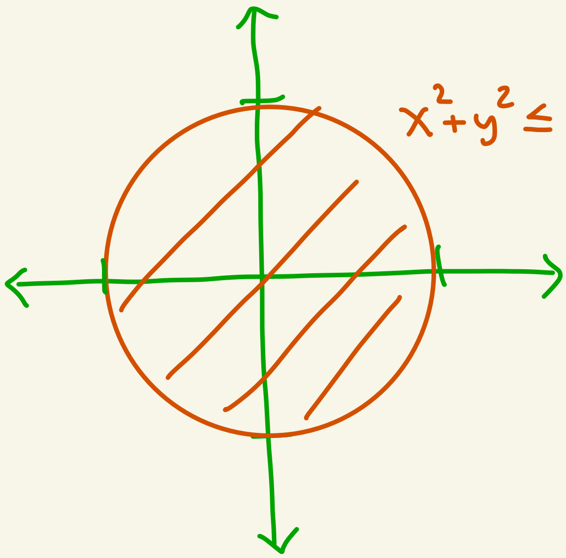
$$r^2 = x^2 + y^2 = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} + \frac{2}{4} = 1$$

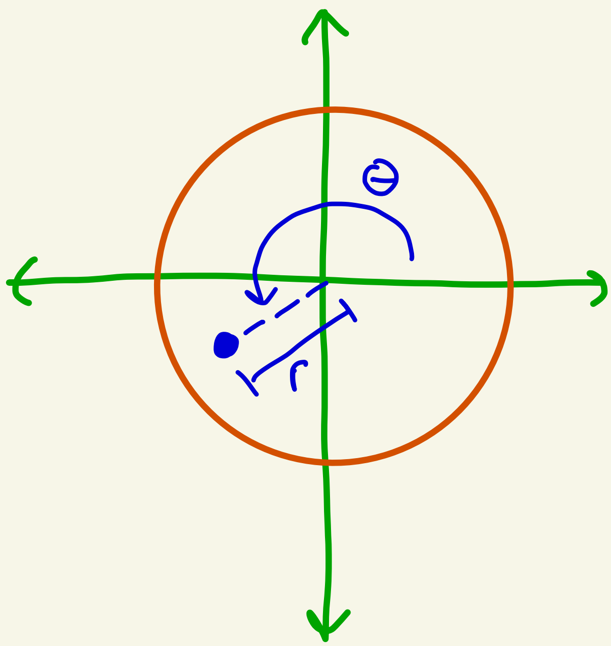
$$\boxed{r = 1}$$

$$\tan(\theta) = \frac{y}{x} = \frac{(\sqrt{2}/2)}{(\sqrt{2}/2)} = 1.$$

$$\boxed{\theta = \pi/4}$$

Ex: Use polar coordinates to describe all points (x,y) satisfying $x^2 + y^2 \leq 1$.

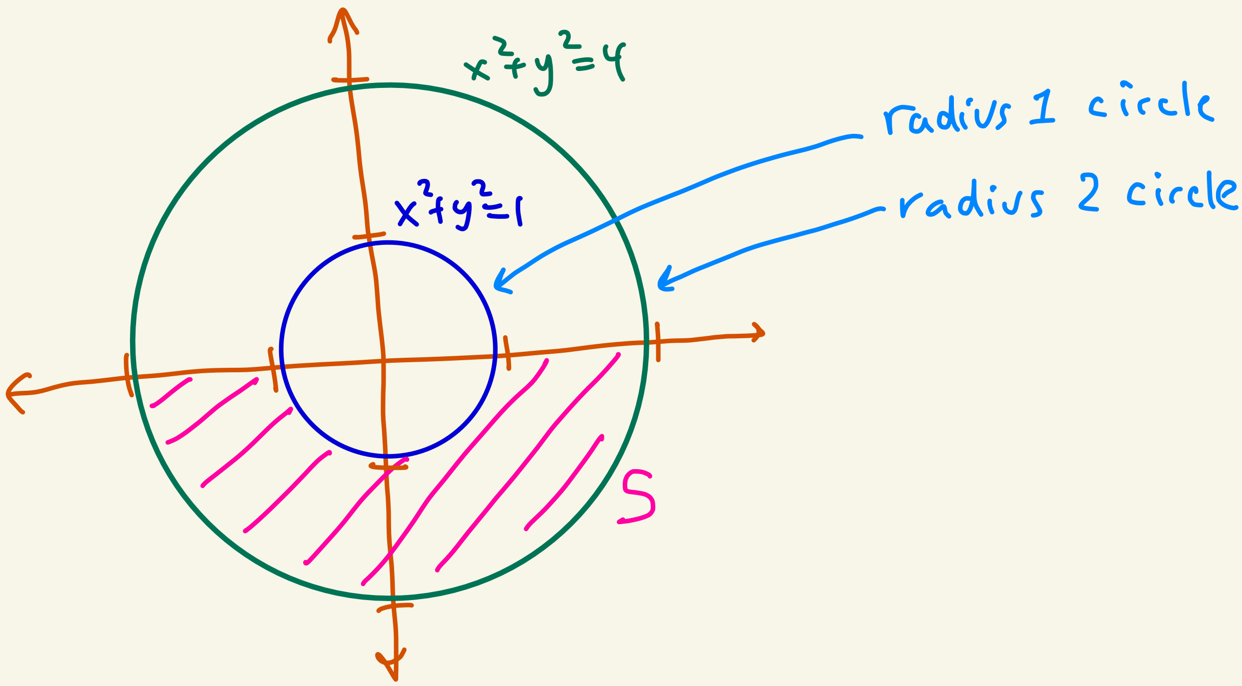


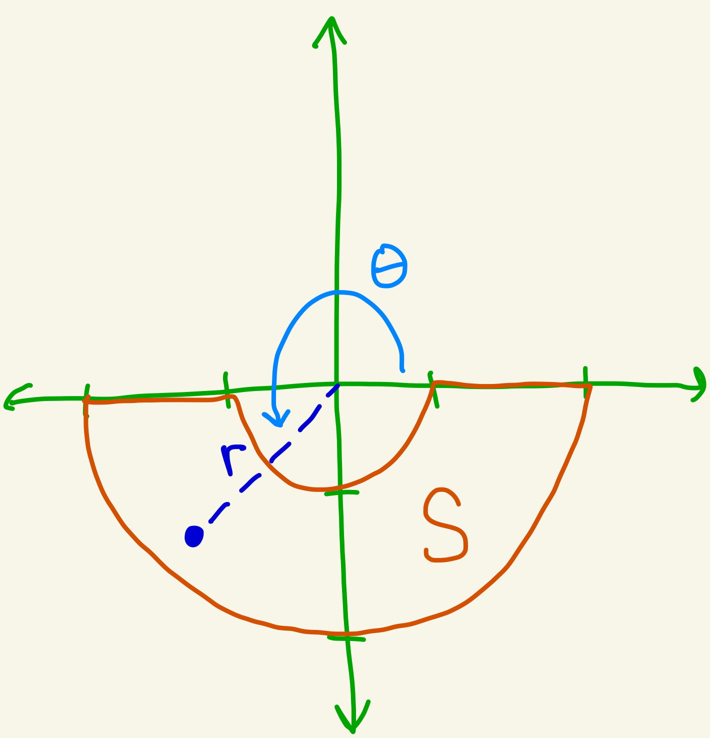


The set is described by

$$0 \leq r \leq 1 \quad \text{and} \quad 0 \leq \theta \leq 2\pi$$

Ex: Use polar coordinates to describe the region S that is below the x-axis and between the circles $x^2+y^2=1$ and $x^2+y^2=4$.



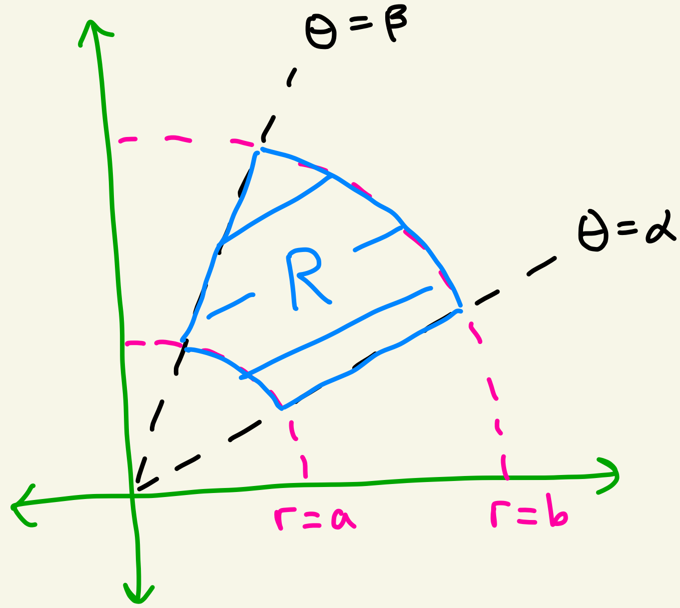


Answer:

$$\pi \le \theta \le 2\pi$$
$$1 \le r \le 2$$

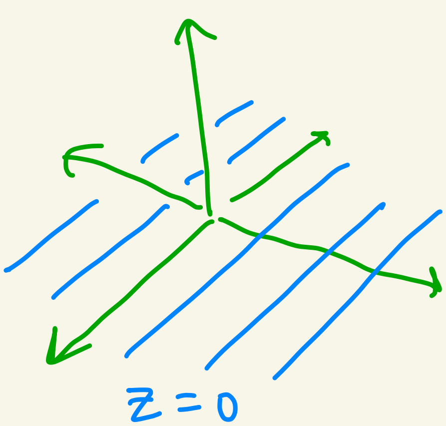
Theorem: (polar coordinate substitution)

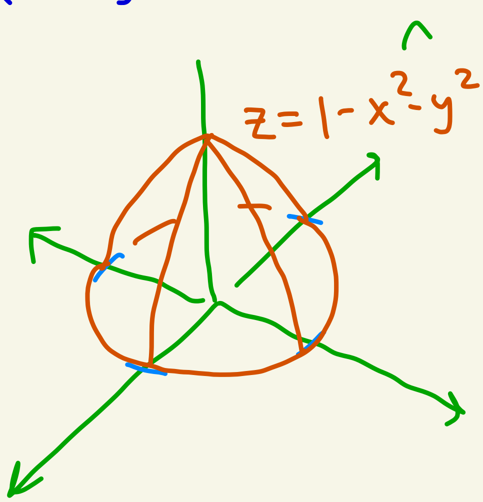
If f is continuous on a "polar" rectangle R given by $0 \le a \le r \le b$ and $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

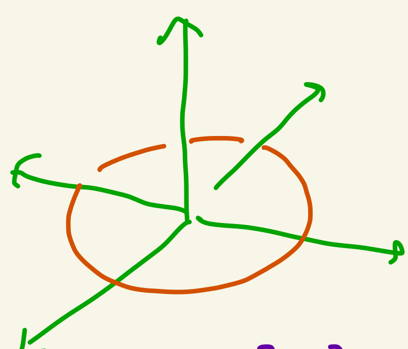


$$\iint_R f(x,y)\,dA = \underbrace{\int_\alpha^\beta}_{\theta \text{ bounds}} \underbrace{\int_a^b}_{r \text{ bounds}} f(\underbrace{r\cos(\theta)}_{x}, \underbrace{r\sin(\theta)}_{y})\, \underbrace{r\,dr\,d\theta}_{dA}$$

Ex: Find the volume V of the solid S that is bounded by $z = 0$ and $z = 1 - x^2 - y^2$.



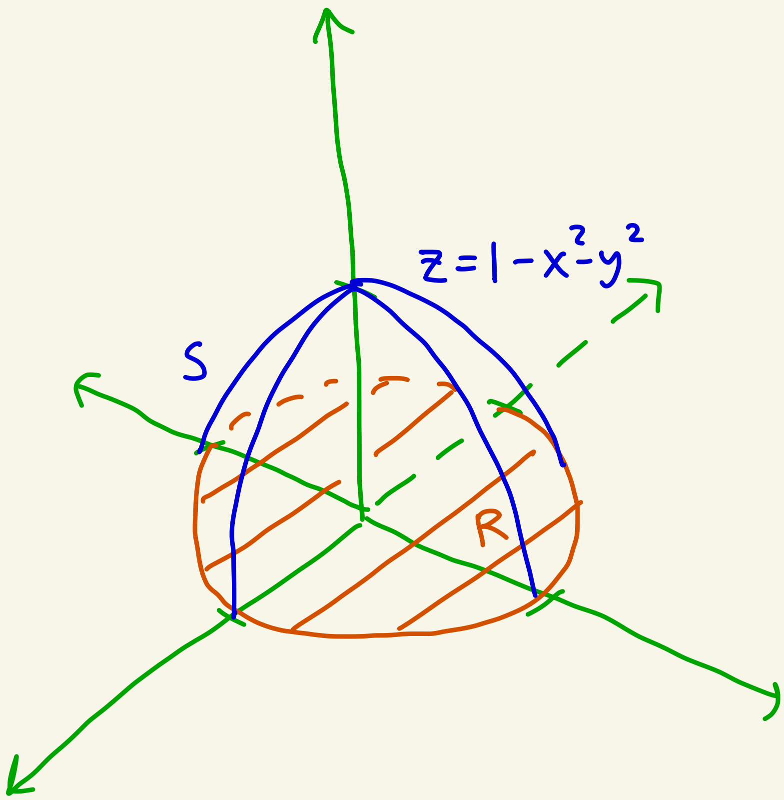


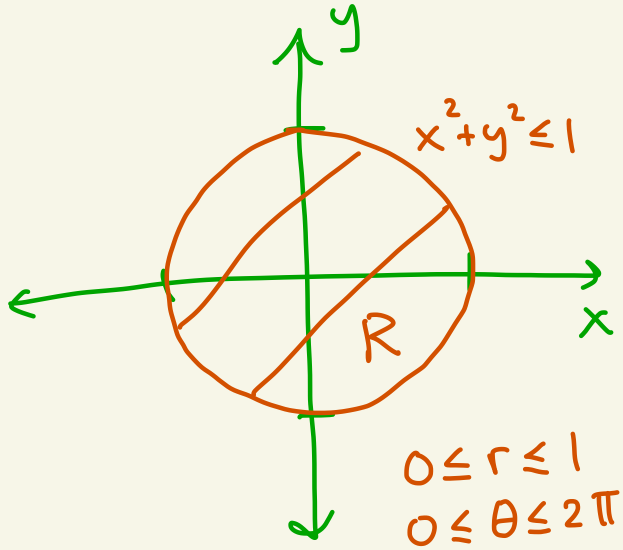


$z = 0$ and $1 - x^2 - y^2$ interrect on this circle where

$0 = 1 - x^2 - y^2$

$x^2 + y^2 = 1$





$0\le r\le 1$
$0\le\theta\le 2\pi$

The volume is

$$\iint_R (1-x^2-y^2)\,dA$$

$$\boxed{\begin{aligned}x&=r\cos(\theta)\\ y&=r\sin(\theta)\\ x^2+y^2&=r^2\\ dA&=r\,dr\,d\theta\end{aligned}}$$

$$=\int_{\underbrace{0}_{\substack{\theta\\ \text{bounds}}}}^{2\pi}\int_{\underbrace{0}_{\substack{r\\ \text{bounds}}}}^{1}\underbrace{(1-r^2)}_{1-(x^2+y^2)}\,\underbrace{r\,dr\,d\theta}_{dA}$$

$$=\int_0^{2\pi}\int_0^1 (r-r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^1 (r - r^3)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2} r^2 - \frac{1}{4} r^4 \Big|_0^1 \right] d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2} - \frac{1}{4} - 0 \right] d\theta$$

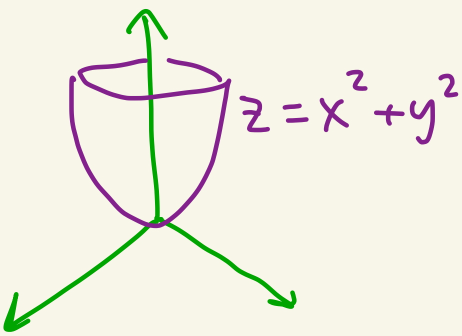
$$= \int_0^{2\pi} \frac{1}{4}\, d\theta$$

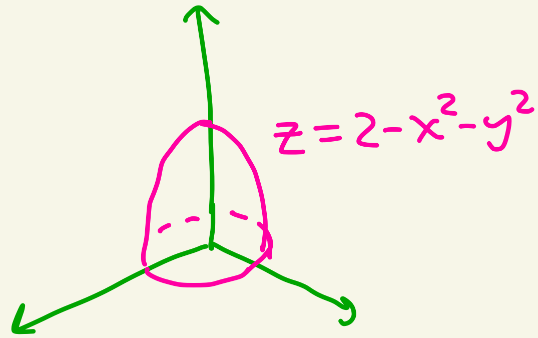
$$= \frac{1}{4}\theta \Big|_0^{2\pi}$$

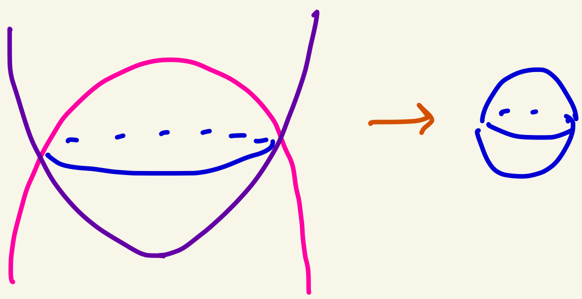
$$= \frac{1}{4}(2\pi - 0)$$

$$= \boxed{\pi/2}$$

Ex: Compute the volume of the solid that lies between the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$.

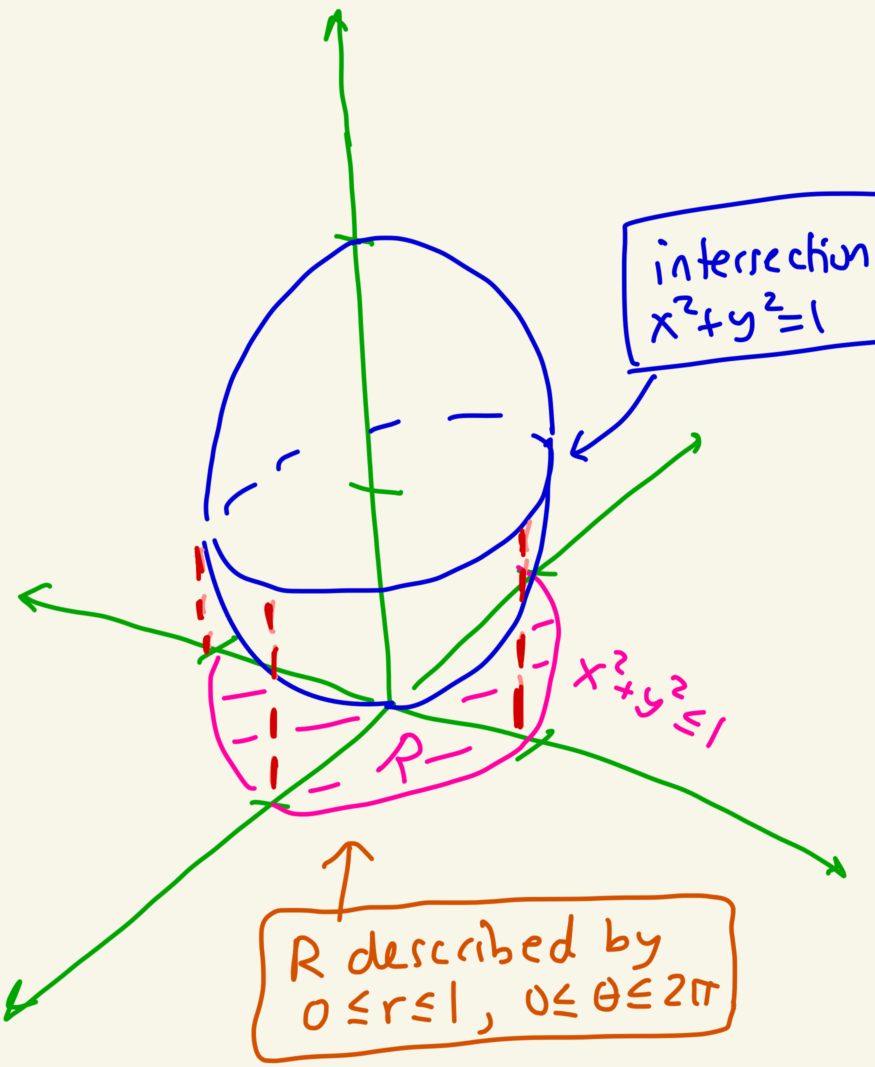






Paraboloids intersect when:

$$x^2 + y^2 = 2 - x^2 - y^2$$
$$2x^2 + 2y^2 = 2$$
$$x^2 + y^2 = 1$$



$$\text{Volume} = \iint\limits_R \Big[\underbrace{(2 - x^2 - y^2) - (x^2 + y^2)}_{\text{top surface} - \text{bottom surface}} \Big] \, dA$$

$x^2 + y^2 = r^2$

$dA = r\,dr\,d\theta$

$0 \le r \le 1$

$0 \le \theta \le 1$

$$= \int_0^{2\pi} \int_0^1 \Big[(2 - r^2) - (r^2) \Big] r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (2r - 2r^3) \, dr \, d\theta$$

$$= \int_0^{2\pi} \Big[r^2 - \frac{2}{4} r^4 \Big]_0^1 \, d\theta$$

$$= \int_0^{2\pi} \Big[\left(1 - \frac{1}{2}\right) - (0 - 0) \Big] \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \, d\theta$$

$$= \frac{1}{2} \theta \Big|_0^{2\pi}$$

$$= \frac{1}{2}(2\pi - 0) = \boxed{\pi}$$